Linear Models Summer 2011
Solutions

Problem 1

(1) The total matrix for $\lambda_1 u_1 + \lambda_2 u_2 = u_3$ is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ -1 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_1 \\ \\ \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} R_2 = \frac{1}{2}R_2 \\ R_3 = R_3 - \frac{1}{2}R_2 \\ R_4 = R_3 - \frac{1}{2}R_2 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so u_1 and u_2 are lin. indep. and $u_1 - 2u_2 = u_3$.
Hence $u_3 \in \operatorname{span}\{u_1, u_2\}$, so $\operatorname{span}\{u_1, u_2, u_3\} = \operatorname{span}\{u_1, u_2\}$.

(2) $u_3 = (1, -2)$

(3) $Tu_1 = (1,1)$, $Tu_3 = T(u_1 - 2u_2) = Tu_1 - 2Tu_2 = u_1 - u_2$

hence $u_1 + u_2 - 2Tu_2 = u_1 - u_2 \iff -2Tu_2 = -2u_2$

$\iff Tu_2 = u_2$

so $Tu_2 = (0,1)$

$$T \sim \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(4) $\det(T) = 1$ so T is invertible, $T^{-1} \sim \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(5) $Tu_2 = u_2$ so u_2 is an eigenvector with corresp. eigenvalue $\lambda = 1$.

Problem 2

(1) From $D = Q^T A Q \Leftrightarrow A = Q D Q^T$ we find

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(2) In (1), replace D with $\ln(D) = \begin{bmatrix} \ln(1) & 0 & 0 \\ 0 & \ln(2) & 0 \\ 0 & 0 & \ln(3) \end{bmatrix}$

and find

$$\ln(A) = \begin{bmatrix} \frac{1}{2}(\ln(1)+\ln(3)) & 0 & \frac{1}{2}(\ln(1)-\ln(3)) \\ 0 & \ln(2) & 0 \\ \frac{1}{2}(\ln(1)-\ln(3)) & 0 & \frac{1}{2}(\ln(1)+\ln(3)) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\ln(3) & 0 & -\frac{1}{2}\ln(3) \\ 0 & \ln(2) & 0 \\ -\frac{1}{2}\ln(3) & 0 & \frac{1}{2}\ln(3) \end{bmatrix}$$

(3) Since $\ln(1) = 0$ is an eigenvalue, $\ln(A)$ is not invertible.

Problem 3

(1) $$\int \cos^2(2x)\sin(3x)\,dx = \int \left(\frac{e^{i2x}+e^{-i2x}}{2}\right)^2\left(\frac{e^{i3x}-e^{-i3x}}{2i}\right)dx$$

$$= \int \frac{1}{4}\left(e^{i4x}+e^{-i4x}+2\right)\frac{e^{i3x}-e^{-i3x}}{2i}\,dx$$

$$= \int \frac{1}{4}\left(e^{i7x}+e^{-ix}+2e^{i3x}-e^{+ix}-e^{-i7x}-2e^{-i3x}\right)\frac{1}{2i}\,dx$$

$$= \int \frac{1}{4}\left(e^{i7x}-e^{-i7x}+2\left(e^{i3x}-e^{-i3x}\right)-\left(e^{ix}-e^{-ix}\right)\right)\frac{1}{2i}\,dx$$

$$= \frac{1}{4}\int\left(\sin(7x)+2\sin(3x)-\sin(x)\right)dx$$

$$= -\frac{1}{4}\left(\frac{1}{7}\cos(7x)+\frac{2}{3}\cos(3x)-\cos(x)\right)+k$$

$$= -\frac{1}{28}\cos(7x)-\frac{1}{6}\cos(3x)+\frac{1}{4}\cos(x)+k.$$

(2) $$2z^2-4z+4=0$$

$$z = \frac{4\pm\sqrt{16-32}}{4} = \frac{4\pm i4}{4} = \underline{\underline{1\pm i}}$$

Problem 4

(1) $\left|\frac{2}{3-\cos(x)}\right| < 1 \iff 2 < 3-\cos(x)$

$\iff \cos(x) < 1$

$\iff x \neq p\cdot 2\pi \;,\; p \in \mathbb{Z}$.

(2) $f(x) = \dfrac{1}{1-\dfrac{2}{3-\cos(x)}} = \dfrac{3-\cos(x)}{1-\cos(x)} \;,\; x \neq p\cdot 2\pi$

(3) For $x \to p\cdot 2\pi$, $f(x) \to \infty$, and

$\min(f) = 3$, so $R(f) = [3, \infty[$.